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Cofinitely soc-supplemented lattices

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Abstract

In this paper, we define Rad-supplemented lattice and its dual soc-supplemented lattice. Furthermore, we show that in compactly generated lattices the first one is equivalent to supplemented lattice. Also, we study some properties of soc-supplemented lattices. Finally, we define cofinitely soc-supplemented lattice (briefly css-lattice) and proved that an arbitrary join of css-lattices is a css-lattice.

Introduction

Throughout this paper L denotes an arbitrary complete modular bounded lattice with smallest element 0 and greatest element 1. A subset S of a lattice L is called a sublattice if it is closed under the operations $\wedge$ and $\vee$. A quotient sublattice b/a for $a \le b$ represents the sublattice $\{x \in L \mid a \le x \le b\}$ (see [3]). An element a is called small in L and denoted by $a \ll L$ if $a \vee b \neq 1$ holds for every $b \neq 1$. An element c is called a supplement of b in L if it is minimal relative to the property $b \vee c = 1$, equivalently $b \vee c = 1$ and $b \wedge c \ll c/0$. A lattice L is called supplemented if each element of L has a supplement in L (see [1]). An element c of a lattice L is called compact if for every subset X of L and $c \le \bigvee X$ there is a finite subset $F \subseteq X$ such that $c \le \bigvee F$. A lattice is called compact if the element 1 is compact. An element a of a lattice L is called cofinite if the quotient sublattice $1/a$ is compact. A lattice L is called cofinitely supplemented if every cofinite element of L has a supplement in L. An element $b \in L$ is called a complement of a if $a \wedge b = 0$ and $a \vee b = 1$. A lattice L is called compactly generated if each element of L is a join of compact elements (see [8]). A subset I of a lattice L is called an ideal in L if I is closed under join and $x \wedge y \in I$ for every $x \in I$ and $y \in L$. An element $m \in L$ is called maximal in L if there is no element greater than m. In a lattice L the meet of all the maximal elements different from 1 in L is called the radical of L, denoted by $r(L)$. An element a in a lattice L is called an atom if there is no element $b \in L$ such that $0 < b < a$. The join of all atoms of L, denoted by $s(L)$, is called the socle of a lattice L (see [3]). A function $f: L \to \acute{L}$ between two lattices L and $\acute{L}$ is called homomorphism if $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in L$. A $\{0,1\}$ −homomorphism is a homomorphism such that $f(0) = 0$ and $f(1) = 1$.

Rad-supplemented lattices and soc-supplemented lattices

Let L be a lattice. Then L is a rad-supplemented (dually soc-supplemeented) lattice if for any $a \in L$, there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b \le r({}^{b}/_{0})$ ($a \wedge b \le s({}^{b}/_{0})$).

In compactly generated lattices, $r(L)$ is the join of all small elements of L and as the set of all small elements form an ideal, so supplemented lattices and rad-supplemented lattices are equivalent (see [8]).

Lemma 2.1. Let L be a compactly generated lattice and $a \in L$. Then $s(L) \wedge a = s({}^{a}/_{0})$.

Proof. Let $s(L) = \bigvee_{i \in J} s_i$ and $s({}^{a}/_{0}) = \bigvee_{i \in I} a_i$ where s_i, a_i are atoms of L and ${}^{a}/_{0}$ respectively. Then $s(L) \wedge a = \bigvee_{i \in J} s_i \wedge a = \bigvee_{i \in J}(s_i \wedge a)$. Since $s_i \wedge a = \begin{cases} a_j & s_i = a_j \; for\ some\ j \\ 0 & s_i \neq a_j \; otherwise \end{cases}$

So, $s(L) \wedge a = s({}^{a}/_{0})$.

From the above lemma, we have $a \wedge b \leq s({}^{b}/_{0}) = s(L) \wedge b \leq s(L)$ and $a \wedge b \leq s(L)$ implies $a \wedge b \leq s(L) \wedge b = s({}^{b}/_{0})$. Hence we can restate the condition $a \wedge b \leq s({}^{b}/_{0})$ in the definition of soc-supplemented lattice by $a \wedge b \leq s(L)$.

Lemma 2.2. If s is a nonzero atom in L, then $s \vee a$ is an atom in ${}^{1}/_{a}$, for all $a \in L$.

Proof. If $s \vee a \neq a$, then $\frac{s \vee a}{a} \cong \frac{s}{s \wedge a} = \frac{s}{0}$. Hence $s \vee a$ is an atom in ${}^{1}/_{a}$.

We conclude from the above lemma that for any lattice L and $a \in L$, $s(L) \vee a = s({}^{1}/_{a})$. Now, by using this fact we obtain the following proposition.

Proposition 2.3. If L is a soc-supplemented lattice, then for every element a, the quotient sublattice ${}^{1}/_{a}$ is also soc-supplemented lattice.

Proof. Let $b \in \frac{1}{a}$. Then there exists a soc-supplement element x of b in L, that is $x \vee b = 1$ and $x \wedge b \leq s(L)$. It is clear that $(a \vee x) \vee b = 1$. Now $(a \vee x) \wedge b = a \vee (x \wedge b) \leq a \vee s(L) = a \vee (\vee x_i) = \vee (a \vee x_i) = s({}^{1}/_{a})$.

Lemma 2.4. Let ${}^{1}/_{a}$ be a soc-supplemented sublattice of a lattice L with $a \leq s(L)$. Then L is a soc-supplemented lattice.

Proof. For all $x \in L$, $x \vee a \in {}^{1}/_{a}$, so there exists $y \in {}^{1}/_{a}$ such that $(x \vee a) \vee y = 1$ and $y \wedge (x \vee a) \leq s({}^{1}/_{a})$. Since $a \leq y$, $x \vee y = x \vee (a \vee y) = 1$. Then $a \wedge (x \wedge y) \leq a \leq s(L)$ implies that L is a soc-supplemented lattice.

In the module theory, every semisimple module is soc-supplemented. The same is true for lattices.

Lemma 2.5. Let L be a lattice and $s(L) = \vee s_i$. Then ${}^{s(L)}/_{0}$ is a soc-supplemented lattice.

Proof. For all $a \in {}^{s(L)}/_{0}$, if a is an atom, then there exists $b = \vee_{s_i \neq a} s_i$ such that $a \vee b = s(L)$ and $a \wedge b \leq s(L) = s(\frac{s(L)}{0})$. If a is not an atom, then there exists $b = \vee_{s_i \neq c} s_i$ where c is an atom in ${}^{a}/_{0}$ such that $a \vee b = s(L)$ and $a \wedge b \leq s(L) = s(\frac{s(L)}{0})$.

Proposition 2.6. Let a be a soc-supplement of an element in a soc-supplemented lattice L. Then ${}^{a}/_{0}$ is a soc-supplemented lattice.

Proof. Let a be a soc-supplement of b in L, so $a \vee b = 1$ and $a \wedge b \leq s({}^{a}/_{0})$. By Proposition 2.3, ${}^{1}/_{b}$ is a soc-supplemented lattice and as $\frac{1}{b} = \frac{a \vee b}{b} \cong \frac{a}{a \wedge b}$ is a soc-supplemented lattice. Hence by Lemma 2.4, we have ${}^{a}/_{0}$ is a soc-supplemented lattice.

Proposition 2.7. Let a be a soc-supplement element of b in L and $s \leq s(L)$. Then a is a soc-supplement of $b \vee s$ in L.

Proof. By assumption, we have $a \vee b = 1$ and $a \wedge b \leq s(L)$. Clearly, $a \vee (b \vee s) = 1$ and $a \wedge (b \vee s) = (a \wedge b) \vee (a \wedge s) \leq s(L) \vee s = s(L)$.

Proposition 2.8. Let a and b be two elements of a soc-supplemented lattice L with $a \vee b = 1$. Then a has a soc-supplement x in L with $x \leq b$.

Proof. Since $a \wedge b$ in L and L is a soc-supplemented lattice, there exists c in L such that $(a \wedge b) \vee c = 1$ and $a \wedge b \wedge c \leq s(L)$. Note that $b = b \wedge 1 = b \wedge [(a \wedge b) \vee c] = (a \wedge b) \vee (b \wedge c)$,
$1 = a \vee b = a \vee [(a \wedge b) \vee (b \wedge c)] = a \vee (b \wedge c)$. Therefore, $x = b \wedge c$ is a soc-supplement of a in L.

Lemma 2.9. Let a be in L such that ${}^{a}/_{b}$ is soc-supplemented for some $b \in L$ with $s({}^{1}/_{b}) \leq s(L)$. If k in L such that $a \vee k$ has a soc-supplemented in L, then k also has a soc-supplement in L.

Proof. By assumption, $a \vee k$ has a soc-supplement of x in L. That is $(a \vee k) \vee x = 1$ and $(a \vee k) \wedge x \leq s(L)$. We consider two cases. First suppose that $a \wedge (k \vee x) \leq b \leq s(L)$. Then $(a \vee x) \wedge k \leq \big(a \wedge (k \vee x)\big) \vee (x \wedge (a \vee k)) \leq s(L)$. Note that $(a \vee x) \vee k = 1$, $a \vee x$ is a soc-supplement of k in L. On the other hand, if $a \wedge (k \vee x) \nleq b$, then we may choose a soc-supplement y of $b \vee [a \wedge (k \vee x)]$ in ${}^{a}/_{b}$. Hence $a = y \vee [a \wedge (k \vee x)]$ and $y \wedge [b \vee \big(a \wedge (k \vee x)\big)] \leq s({}^{a}/_{b}) \leq s({}^{1}/_{b}) \leq s(L)$ and so $y \wedge (k \vee x) \leq y \wedge [b \vee (k \vee x)] = y \wedge [b \vee \big(a \wedge (k \vee x)\big)] \leq s(L)$. Hence $1 = a \vee x \vee k = [y \vee (a \wedge (k \vee x)] \vee (x \vee k) = y \vee x \vee k$ and $(y \vee x) \wedge k \leq (y \wedge k) \vee (x \wedge k) \leq [y \wedge (k \vee x)] \vee [x \wedge (y \vee k)] \leq [y \wedge (k \vee x)] \vee [x \wedge (a \vee k)] \leq s(L)$.

The following theorem shows that the join of two soc-supplemented sublattices of a lattice is soc-supplemented.

Theorem 2.10. Let a_1 and a_2 be two elements in a lattice L such that $a_1 \vee a_2 = 1$ and ${}^{a_1}/_{0}$, ${}^{a_2}/_{0}$ are soc-supplemented lattices. Then L is soc-supplemented lattice.

Proof. For any x in L, we have $a_1 \vee (a_2 \vee x) = 1$. Since 0 is a soc-supplemented of $a_1 \vee (a_2 \vee x)$ in L, by Lemma 2.9, $a_2 \vee x$ has a soc-supplement in L. Applying Lemma 2.9 again we obtain a soc-supplement for x.

Theorem 2.11. Let ${}^{a}/_{0}$ and ${}^{1}/_{a}$ be soc-supplemented sublattices of a lattice L and a has a soc-supplement in L. Then L is soc-supplemented lattice.

Proof. Let b be the soc-supplemented of a in L. Then $a \vee b = 1$ and $a \wedge b \leq s(L)$. As ${}^{a}/_{0}$ is a soc-supplemented, ${}^{a}/_{a \wedge b}$ is soc-supplemented sublattice of ${}^{a}/_{0}$ by Proposition 2.3. Also, since ${}^{b}/_{a \wedge b} \cong {}^{a \vee b}/_{a} \cong {}^{1}/_{a}$, ${}^{b}/_{a \wedge b}$ is soc-supplemented quotient sublattice of ${}^{1}/_{a}$. Now, ${}^{1}/_{a \wedge b} = {}^{a \vee b}/_{a \wedge b} \cong \left[{}^{a}/_{a \wedge b}\right] \vee \left[{}^{b}/_{a \wedge b}\right]$ is soc-supplemented by Theorem 2.10. Hence L is soc-supplemented lattice by Lemma 2.4.

A lattice L is called complemented if every element a of L is a complement of an element b in L, that is $a \vee b = 1$ and $a \wedge b = 0$ for some b in L. It is clearly that if L is complemented lattice, then ${}^{a}/_{0}$ is complemented for every element a of L (see [5]). An element a of L is called essential if $a \wedge b \neq 0$ for every nonzero element b in L and denoted by $a \trianglelefteq L$. It is easy to see that if $a \trianglelefteq L$, then for every element b of L, $a \wedge b \trianglelefteq {}^{b}/_{0}$ (see [6] and [4, Exercise 4.5]). An element b is said to be E −complement of an element a of L if $a \wedge b = 0$ and $a \vee b \trianglelefteq L$. A lattice L is said to be E −complemented if every element of L has an E −complement in L (see [7]).

Lemma 2.12. Let L be an E −complemented lattice and a be an element of L different form 0,1. If the quotient sublattice ${}^{1}/_{a}$ is complemented, then there exist b_1, b_2 in L such that b_1 is complement of b_2, ${}^{b_1}/_{0}$ is complemented, $a \trianglelefteq {}^{b_2}/_{0}$ and ${}^{b_2}/_{a}$ is complemented.

Proof. By E −complementedness of L, we can find an element $b_1 \in L$ such that $b_1 \wedge a = 0$ and $b_1 \vee a \trianglelefteq L$. As ${}^{1}/_{a}$ is complemented, there is a complement b_2 of $b_1 \vee a$ in ${}^{1}/_{a}$. Then $1 = (b_1 \vee a\,) \vee b_2 = b_1 \vee b_2$ and $0 = b_1 \vee a = b_1 \wedge [(b_1 \vee a\,) \wedge b_2] = b_1 \wedge [(b_1 \wedge b_2) \vee a] = (b_1 \wedge b_2) \vee (b_1 \vee a\,) = b_1 \wedge b_2$. Also, since

${}^{1}/_{a}$ is complemented, ${}^{b_2}/_{a}$ and ${}^{b_1}/_{0} = {}^{b_1}/_{(b_1 \wedge a)} \cong {}^{(b_1 \vee a)}/_{a}$ are complemented. As $b_1 \vee a \trianglelefteq L$, $a = (b_1 \wedge b_2) \vee a = (b_1 \vee a) \wedge b_2 \trianglelefteq {}^{b_2}/_{0}$.

Lemma 2.13. Let L be an E −complemented lattice and a be an element of L different from 0,1. The quotient sublattice ${}^{1}/_{a}$ is complemented if and only if for every element b of L, there exists an element c in L such that $b \vee c = 1$ and $b \wedge c \leq a$.

Proof. Let ${}^{1}/_{a}$ be complemented and b in L. Since $b \vee a$ in ${}^{1}/_{a}$, it has a complement c in ${}^{1}/_{a}$. Then $(b \wedge c) \vee a = (b \vee a) \wedge c = a$. Hence $b \wedge c \leq a$ and $b \vee c = (b \vee a) \vee c = 1$.

Conversely, take b in ${}^{1}/_{a}$. Then there is an element c of L with $b \vee c = 1$ and $b \wedge c \leq a$. Thus $b \vee (c \vee a) = b \vee c = 1$ and $b \wedge (c \vee a) = (b \wedge c) \vee a = a$ and this means that $c \vee a$ is a complement of b in ${}^{1}/_{a}$. Therefore ${}^{1}/_{a}$ is complemented.

A lattice L is called soc-complemented if the quotient sublattice ${}^{1}/_{s(L)}$ is complemented. Now, we use the above two lemmas to give the following main result.

Theorem 2.14. If L is an E −complemented soc-supplemented lattice, then it is soc-complemented and there are elements b_1, b_2 in L such that b_1 is a complement of b_2 with ${}^{b_1}/_{0}$ complemented and $s(L) \trianglelefteq {}^{b_2}/_{0}$.

Proof. Since L is a soc-supplemented, for all b in L there exists c in L such that $b \vee c = 1$ and $b \wedge c \leq s(L)$. By taking $a = s(L)$ in Lemma 2.13, we obtain ${}^{1}/_{s(L)}$ is complemented. Then by Lemma 2.12, there exist b_1, b_2 in L such that b_1 is complement of b_2, ${}^{b_1}/_{0}$ is complemented and so $s(L) \trianglelefteq {}^{b_2}/_{0}$.

Lemma 2.13, tells us that if a lattice L is soc-complemented, then it is soc-supplemented lattice. Therefore, this fact and theorem 2.14 together give the following corollary.

Corollary 2.15. Let L be an E −complemented lattice. Then L is soc-supplemented if and only if it is soc-complemented.

Let L and $\acute{L}$ be two lattices and $f: L \to \acute{L}$ be a $\{0,1\}$ −homomorphism. Since any homomorphism between lattices preserves the operations join and meet, we have $f(s(L)) \leq s(f(L))$. The following proposition generalize [2, Proposition 2.15].

Proposition 2.16. Any $\{0,1\}$ −homomorphic image of soc-supplemented lattice is soc-supplemented.

Proof. Let $f: L \to \acute{L}$ be an onto $\{0,1\}$ −homomorphism and L be soc-supplemented lattice. For $x \in \acute{L}$, there exists $a \in L$ such that $x = f(a)$ and as L is soc-supplemented, there exists $b \in L$ with $a \vee b = 1$ and $a \wedge b \leq s(L)$. Now, $1 = f(1) = f(a \vee b) = f(a) \vee f(b) = x \vee f(b)$ and $x \wedge f(b) = f(a) \wedge f(b) = f(a \wedge b) \leq f\big(s(L)\big) \leq s\big(f(L)\big) = s(\acute{L})$. Hence $f(b)$ is a soc-supplement of x in $\acute{L}$ and therefore $\acute{L}$ is soc-supplemented lattice.

3. Cofinitely soc-supplemented lattices

A lattice L is called cofinitely soc-supplemented lattice (briefly css-lattice) if every cofinite element of L has a soc-supplement in L. Now, we have the following lemma.

Lemma 3.1. Let L be a compactly generated lattice and a be a cofinite element of L. If b is a soc-supplement of a in L, then a has a soc-supplement c in L such that $x \leq b$ and x is compact.

Proof. As L is compactly generated, $b = \bigvee_{i \in I} s_i$ where each c_i is compact. Thus $1 = a \vee b = a \vee \bigvee_{i \in I} s_i = \bigvee_{i \in I} a \vee s_i$. Since ${}^{1}/_{a}$ is compact, $1 = \bigvee_{i \in S} (a \vee s_i)$ for some finite subset S of I. Then by [4, Proposition 2.1], $x = \bigvee_{i \in S} s_i$ is compact and hence it is soc-supplement of a.

Proposition 3.2. Let L be css-lattice. Then for every atom a of L, ${}^{1}/_{a}$ is a css-sublattice of L.

Proof. Let b be a cofinite element of ${}^{1}/_{a}$. Then ${}^{1}/_{b}$ is compact sublattice in ${}^{1}/_{a}$, so ${}^{1}/_{b}$ is compact quotient sublattice in L. This implies that b is a cofinite element of L. As L is css-lattice, b has a soc-supplement x in

L, that is, $b \vee x = 1$ and $b \wedge x \leq s(L)$. Then by Lemma 2.2, $b \wedge (x \vee a) = (b \wedge x) \vee a \leq s(L) \vee a = s(^{1}/_{a})$ and hence $^{1}/_{a}$ is css-lattice.

Now, we are going to prove that a lattice L is css-lattice if and only if every maximal element of L has soc-supplement. This result was proved for css-modules in [2]. First, we have the following lemma.

Lemma 3.3. Let a and b be elements of L such that b is a soc-supplement of a maximal element m of L. If $a \vee b$ has a soc-supplemented in L, then a has a soc-supplement in L.

Proof. Let c be a soc-supplement of $a \vee b$ in L. If $b \wedge (a \vee c) \leq b \wedge m \leq s(L)$, then $b \vee c$ is a soc-supplement of a, because $(b \vee c) \wedge a \leq [b \wedge (a \vee c)] \vee [c \wedge (a \vee b)] \leq s(L)$. Let $b \wedge (a \vee c) \nleq b \wedge m$. Since $\frac{b}{b\wedge m} \cong \frac{b\vee m}{m} = \frac{1}{m}$, $b \wedge m$ is maximal element of $^{b}/_{0}$. Thus $(b \wedge m) \vee [b \wedge (a \vee c)] = b$. Hence $a \wedge c \leq (a \vee b) \wedge c \leq s(L)$ and $1 = c \vee a \vee b = c \vee a \vee (b \wedge m) \vee [b \wedge (a \vee c)] = c \vee a$, which means that c is a soc-supplement of a in L. Therefore in both cases there is a soc-supplement of a in L.

Let Γ be the set of all elements b of L such that b is a soc-supplement of some maximal element of L and let $css(L)$ denote the join of all elements of Γ.

Theorem 3.4. A lattice L is a css-lattice if and only if every maximal element of L has a soc-supplement.

Proof. The proof of one side comes from fact that every maximal element is cofinite. For the other side, since in [4, Lemma 2.4] we have every non-zero compact lattice has maximal element. The remain of the proof is analogous to the proof of [2, Theorem 2.15].

Using the above theorem we prove that an arbitrary join of css-lattices is css-lattice (see [2, Theorem 2.12]).

Theorem 3.5. Let $\{^{a_i}/_{0}\}_{i\in I}$ be a collection of css-sublattices of L with $1 = \vee_{i\in I}\, a_i$. Then L is a css-lattice.

Proof. Suppose m is a maximal element of L. If $a_i \leq m$, for all $i \in I$, then $1 = \vee_{i\in I}\, a_i \leq m$, so $m = 1$ which is a contradiction. Hence there exists $k \in I$ such that $a_k \nleq m$ and $1 = a_k \vee m$. As $\frac{a_k}{a_k \wedge m} \cong \frac{a_k \vee m}{m} = \frac{1}{m}$, $a_k \wedge m$ is maximal in $^{a_k}/_{0}$. Let c be soc-supplement of $a_k \wedge m$ in $^{a_k}/_{0}$. If $c \leq m$, then $a_k = (a_k \wedge m) \vee c \leq m$ which is a contradiction. Hence $c \nleq m$ and so $1 = m \vee c$ and $m \wedge c = m \wedge a_k \wedge c \leq s(L)$. Thus c is soc-supplement of m in L and by Theorem 3.4, L is css-lattice.

We use theorem 3.4 to prove the following result.

Theorem 3.6. Let $a \in L$ such that $^{1}/_{a}$ has no maximal element and $^{a}/_{0}$ be css-sublattice of L. Then L is also css-lattice.

Proof. Let m be a maximal element of L. If $a \leq m$, then m is a maximal element of $^{1}/_{a}$, but $^{1}/_{a}$ has no maximal element. So $a \nleq m$. Thus $a \vee m = 1$. Since $\frac{a}{a\wedge m} \cong \frac{a\vee m}{m} = \frac{1}{m}$, $a \wedge m$ is a maximal element of $^{a}/_{0}$ and therefore a cofinite element of $^{a}/_{0}$. So there is a soc-supplement c of $a \wedge m$ in $^{a}/_{0}$, that means $(a \wedge m) \vee c = a$ and $(a \wedge m) \vee c \leq s(^{a}/_{0})$. As c in $^{a}/_{0}$, $c \wedge m = c \wedge (a \wedge m) \leq s(^{a}/_{0}) \leq s(L)$ and $c \vee m = c \vee (a \wedge m) \vee m = a \vee m = 1$. Hence c is a soc-supplement of b in L and by Theorem 3.4, L is css-lattice.

Lemma 3.7. Suppose that a is a cofinite element of a compactly generated css-lattice such that $^{b}/_{0}$ has small socle for every compact b in L. If a has a soc-supplement b in L, then a has compact supplement in L.

Proof. Let a be cofinite element in L. Then $^{1}/_{a}$ is compact and by Lemma 3.1, a has a compact soc-supplement c in L with $c \leq b$, that is $a \vee c = 1$ and $a \wedge c \leq s(^{c}/_{0})$. Since $s(^{c}/_{0}) \ll {}^{c}/_{0}$, c is a compact supplement of a in L.

Finally, we give conditions under which css-lattice and cofinitely supplemented lattices are equivalent.

Theorem 3.8. Let L be compactly generated lattice with the property that $^{a}/_{0}$ has small socle, for every compact element a in L and every small element is a join of atom elements. Then L is css-lattice if and only if L is cofinitely supplemented.

Proof. Let a be cofinite element of L. Since L is css-lattice, a has a soc-supplement b in L and by Lemma 3.7, a has compact supplement. Thus L is cofinitely supplemented. Now, if L is cofinitely supplemented lattice, then for all cofinite element a of L, there exists b in L such that $a \vee b = 1$ and $a \wedge b \ll {}^{b}/_{0}$ and by hypothesis, $a \wedge b \leq s({}^{b}/_{0})$. Hence L is css-lattice.

Corollary 3.9. Let L be a compact lattice such that $s({}^{a}/_{0}) \ll {}^{a}/_{0}$ for every compact element a in L and every small element of L is a join of atom elements. Then L is soc-supplement if and only if it is supplemented.

Proof. In compact lattice L, every element is cofinite, so the proof immediately follows from Theorem 3.8.

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